A METHOD FOR MIDCOURSE GUIDANCE OPTIMIZATION

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Introduction

Owing to initial injection errors, an actual interplanetary trajectory may deviate from the desired nominal, and a correction or a sequence of corrections must be performed in order to attain the desired terminal position. As time goes on, tracking information improves the knowledge of the vehicle's whereabouts resulting in a best estimate of the trajectory, but, of course, the observations employed have errors in them. When a correction sequence is performed, the terminal miss is not fully nulled owing to actuation errors, i.e., imperfect control performance, as well as to the inaccurate knowledge of the vehicle's true position. Such a correction sequence cannot be described in the framework of the usual differential equations of motion because of the random character of the control errors and observation errors. This "noise" cannot be evaluated for any particular instant of time since it is represented in terms of random variables, but it will ordinarily have known statistical properties. So one may introduce the covariance matrices of normally distributed variables to provide a second order statistical measure, and in these terms the effects of the noise may be studied. The elements of the covariance matrices of the deviations and estimation errors then may be regarded as the new state variables of the midcourse guidance problem. The first portion of this report is concerned with deriving the differential equations satisfied by these matrices. The approach to this development is that employed in the papers by Denham and Speyer (Ref. 1) and Striebel and Breakwell (Ref. 2). Processing of observational data will be taken to be according to the linear filtering theory of Kalman and Bucy (Ref. 3).

The midcourse guidance optimization problem may be phrased in terms of minimizing a statistical measure of the propellant consumption for an acceptable terminal dispersion by choice of the coefficients of a linear feedback guidance law. In the treatment to be presented, the coefficients of the law will be taken from an appropriate single impulse correction policy except for a multiplicative factor representing the magnitude of correction which will assume the role of a scalar control variable of the optimization problem. In the remaining portion of the report, we present an optimization method tailored to the particular form of the midcourse guidance problem. The general idea of the method is the determination of a control history by an iterative process which minimizes a Hamiltonian function along the trajectory. Basically, it is a first order method, and is a variant of a technique previously studied in Ref. 4.

The Variational Equations

The equations of motion are given by a system of simultaneous nonlinear ordinary differential equations written in first order form as

$$\dot{x} = f(x, u) \tag{1}$$

where the x-vector components are the state variables and the u-vector components are some control functions.

Eq. (1) is satisfied by the reference or desired trajectory, and, in the context of the midcourse guidance problem, this will ordinarily consist of free-fall motion from injection to a specified terminal point. The problem of determining a suitable reference trajectory may be regarded as separate from the midcourse correction problem and will not be discussed here.

The variational equations for this system are

$$\delta\dot{x} = F\,\delta x + G\,\delta u + v \tag{2}$$

where $F = \frac{\partial f}{\partial x}$, $G = \frac{\partial f}{\partial u}$, and v is a column vector representing the mechanization errors (i.e., noise or actuation errors); v is taken as uncorrelated Gaussian noise with zero mean, which may be stated $\mathcal{E}[v(t)] = 0$, $\mathcal{E}[v(\tau)\, v'(t)] = Q(t)\, \delta(t - \tau)$ where $\delta(t - \tau)$ is the Dirac delta function.

In Eq. (2)

$$\delta x \equiv x_{True} - x_{Desired}$$

We define

δx	the true or total deviation from the desired.
$\delta \hat{x} = \hat{x} - x_D$	the estimate of the deviation from the desired or nominal.
$\Delta x = x_{True} - \hat{x}$	the error in the estimate of the deviation.

The covariance matrices X, Y, and P are defined:

$$X = \mathcal{E}[\delta x \;\; \delta x']$$

$$Y = \mathcal{E}[\delta \hat{x} \;\; \delta \hat{x}']$$

$$P = \mathcal{E}[\Delta x \;\; \Delta x']$$

The differential equations for the estimate of the deviations from the nominal trajectory are of the assumed form

$$\delta \dot{\hat{x}} = F\, \delta \hat{x} + G\, \delta u + K(\delta Z - \delta \hat{Z}) \tag{3}$$

where $Z(x, \epsilon)$ is a column vector of measurement data

$$\delta Z = Z(x_{True}, \epsilon) - Z(x_{Desired}, 0)$$

$$\delta Z \cong \frac{\partial Z}{\partial X} \delta X + \epsilon = M\, \delta X + \epsilon$$

$$\delta \hat{Z} = M\, \delta \hat{x}$$

Note that $\delta \hat{Z}$ would equal δZ (to first order) if there were no estimation error and no measurement error.

ϵ is a column vector of the measurement errors statistically described by

$$\mathcal{E}[\epsilon(t)] = 0$$

$$\mathcal{E}[\epsilon(\tau)\, \epsilon'(t)] = R(t)\, \delta(t - \tau)$$

The optimal (minimum variance) gain matrix, K, derived by Kalman and Bucy is

$$K = PM'R^{-1}$$

It should be noted that this is optimal only in the case in which the covariance of the actuation errors is independent of the controls and the state, which is not the case in the midcourse situation. However, K above will be assumed to be close enough to the actual optimum to furnish a suitable approximation.

Everything is now specified, at least statistically, except δu. We now assume that the control deviation will be a linear feedback on the estimate deviation quantities, i.e.,

$$\delta u = - \Lambda\, \delta \hat{x}$$

noting that δx cannot be used because it is never known.

The matrix Λ is the matrix of feedback control gains. We now write

$$\delta\dot{x} = F\,\delta x - G\Lambda\,\delta\hat{x} + v \tag{4}$$

$$\delta\dot{\hat{x}} = F\,\delta\hat{x} - G\Lambda\,\delta\hat{x} + KM\,\Delta x + K\epsilon \tag{5}$$

and since $\Delta x = \delta x - \delta\hat{x}$, we have

$$\Delta\dot{x} = F\,\Delta x - KM\,\Delta x - K\epsilon + v \tag{6}$$

The appearance of the random variables v and ϵ now force us to statistical analysis.

To give us a useful statistical measure, we now introduce the co-variance matrices,

$$P(t) = \mathcal{E}[\Delta x(t)\,\Delta x'(t)] \tag{7}$$

$$Y(t) = \mathcal{E}[\delta\hat{x}(t)\;\delta\hat{x}'(t)] \tag{8}$$

$$B(t) = \mathcal{E}[\delta\hat{x}(t)\;\Delta x'(t)] = 0 \quad \text{if} \quad \mathcal{E}[\delta\hat{x}(t_o)\;\Delta x'(t_o)] = 0 \tag{9}$$

This last equation states that, if the initial estimate and the error in the estimate are uncorrelated, B will then be zero for all time. This can be shown by developing the $\dot{B}$ equation and observing that every term in $\dot{B}$ has B as a factor:

$$\dot{B} = FB + BF' - BM'R^{-1}MP - G\Lambda B \tag{10}$$

The Development of the $\dot{P}$ Equation

$$P(t) = \mathcal{E}[\Delta x(t)\ \Delta x'(t)]$$

$$\dot{P} = \mathcal{E}[\Delta\dot{x}\ \Delta x'] + \mathcal{E}[\Delta x\ \Delta\dot{x}']$$

$$= \mathcal{E}[(F\,\Delta x - KM\,\Delta x - K\epsilon + v)\ \Delta x']$$
$$+\ \mathcal{E}[\Delta x\ (\Delta x' F' - \Delta x' M' K' - \epsilon' K' + v')]$$

$$= (F - KM)\ \mathcal{E}[\Delta x\ \Delta x'] + \mathcal{E}[\Delta x\ \Delta x'](F' - M'K')$$
$$+\ \mathcal{E}[(-K\epsilon + v)\ \Delta x'] + \mathcal{E}[\Delta x\ (-\epsilon' K' + v')] \tag{11}$$

In order to evaluate the last term in Eq. (11), $\mathcal{E}[\Delta x\ (-\epsilon' K' + v')]$, we must first integrate

$$\Delta\dot{x} = (F - KM)\,\Delta x - K\epsilon + v$$

The solution is

$$\Delta x(t) = \Phi(t, t_o)\,\Delta x(t_o) + \int_{t_o}^{t} \Phi(t, \tau)\,[-K(\tau)\,\epsilon(\tau) + v(\tau)]\,d\tau \tag{12}$$

provided that Φ has the properties

$$\dot{\Phi}(t, t_o) = [F(t) - K(t)\,M(t)]\ \Phi(t, t_o) \tag{13}$$

with initial value

$$\Phi(t_o, t_o) = I \tag{14}$$

To show that this solution is valid, one may differentiate it.

$$\Delta\dot{x}(t) = \dot{\Phi}(t,t_o)\,\Delta x(t_o) + \Phi(t,t)\left[-K(t)\,\epsilon(t) + v(t)\right] + \int_{t_o}^{t} \dot{\Phi}(t,\tau)\left[-K(\tau)\,\epsilon(\tau) + v(\tau)\right] d\tau \tag{15}$$

And making use of the relations from (12), (13) and (14) respectively

$$\Phi(t,t_o)\,\Delta x(t_o) = \Delta x(t) - \int_{t_o}^{t} \Phi(t,\tau)\left[-K(\tau)\,\epsilon(\tau) + v(\tau)\right] d\tau$$

$$\dot{\Phi}(t,\tau) = \left[F(t) - K(t)\,M(t)\right]\Phi(t,\tau)$$

$$\Phi(t,t) = I$$

we find we have the same equation for $\Delta\dot{x}$ with which we started. So in order to evaluate $\dot{P}(t)$ we have

$$\begin{aligned}\mathcal{E}\left[\Delta x\,(-\epsilon' K' + v')\right] &= \mathcal{E}\left[\left\{\int_{t_o}^{t} \Phi(t,\tau)\left[-K(\tau)\,\epsilon(\tau) + v(\tau)\right] d\tau\right\}\left\{-\epsilon'(t)K'(t) + v'(t)\right\}\right] \\ &= \int_{t_o}^{t} \Phi(t,\tau)\left[K(\tau)R(\tau)K'(t) + Q(\tau)\right]\delta(\tau - t)\,d\tau\end{aligned} \tag{16}$$

because

$$\mathcal{E}\left[\epsilon(t)\ \epsilon'(\tau)\right] = R(t)\,\delta(t-\tau)$$

$$\mathcal{E}\left[v(t)\ v'(\tau)\right] = Q(t)\,\delta(t-\tau)$$

$$\mathcal{E}\left[\epsilon(t)\ v'(\tau)\right] = 0$$

Now the integral of the Dirac delta function from $-\infty$ to $+\infty$ is 1, but we are integrating from t_o to t, where t is precisely the midpoint of this particular function so the value of the integral $\int_{t_o}^{t} \delta(\tau - t)\, d\tau = \frac{1}{2}$ and since $\Phi(t, t) = I$

$$\mathcal{E}[\Delta x(-\epsilon' K' + v')] = \frac{1}{2}[K(t)R(t)K'(t) + Q(t)] \tag{17}$$

Similarly, we find that

$$\mathcal{E}[(-K\epsilon + v)\,\Delta x'] = \frac{1}{2}[K(t)R(t)K'(t) + Q(t)] \tag{18}$$

By substituting equations (7), (17), and (18) into Eq. (11), the expression for $\dot{P}$ becomes

$$\dot{P} = FP - KMP + PF' - PM'K' + KRK' + Q \tag{19}$$

and by making use of the value of $K = PM'R^{-1}$, Eq. (19) reduces to

$$\dot{P} = FP + PF' - PM'R^{-1}MP + Q \tag{20}$$

The Development of the $\dot{Y}$ Equation

$$Y = \mathcal{E}[\delta\hat{x}\ \delta\hat{x}'] \tag{21}$$

We will use the differential equation

$$\delta\dot{\hat{x}} = (F - G\Lambda)\delta\hat{x} + KM\Delta x + K\epsilon \tag{22}$$

and its solution which is

$$\delta\hat{x} = \Psi(t, t_o)\,\delta\hat{x}(t_o) + \int_{t_o}^{t} \Psi(t, \tau)[K(\tau)M(\tau)\Delta x(\tau) + K(\tau)\epsilon(\tau)]\,d\tau \tag{23}$$

where

$$\dot{\Psi}(t, t_o) = [F(t) - G(t)\Lambda(t)]\,\Psi(t, t_o) \tag{24}$$

$$\Psi(t_o, t_o) = I \tag{25}$$

Differentiating Y, we have

$$\begin{aligned}\dot{Y} &= \mathcal{E}[\delta\dot{\hat{x}}\ \ \delta\hat{x}'] + \mathcal{E}[\delta\hat{x}\ \ \delta\dot{\hat{x}}'] \\ &= (F - G\Lambda)\,\mathcal{E}[\delta\hat{x}\ \ \delta\hat{x}'] + \mathcal{E}[\delta\hat{x}\ \ \delta\hat{x}'](F' - \Lambda'G') + KM\,\mathcal{E}[\Delta x\ \ \delta\hat{x}'] \\ &\quad + \mathcal{E}[\delta\hat{x}\ \ \Delta x']M'K' + \mathcal{E}[K\epsilon\ \ \delta\hat{x}'] + \mathcal{E}[\delta\hat{x}\ \ \epsilon'K']\end{aligned} \tag{26}$$

Using Eq. (23) for $\delta\hat{x}$, we find that

$$\begin{aligned}\mathcal{E}[\delta\hat{x}\ \ \epsilon'K'] &= \mathcal{E}\left[\left\{\int_{t_o}^{t} \Psi(t, \tau)\,K(\tau)\,\epsilon(\tau)\,d\tau\right\}\ \epsilon'(t)K'(t)\right] \\ &= \int_{t_o}^{t} \Psi(t, \tau)\,K(\tau)R(\tau)K'(t)\ \delta(\tau - t)\,d\tau \\ &= \frac{1}{2}K(t)R(t)K'(t)\end{aligned} \tag{27}$$

Similarly

$$\mathcal{E}[K(t)\,\epsilon(t)\ \ \delta\hat{x}'(t)] = \frac{1}{2}K(t)R(t)K'(t) \tag{28}$$

Finally, remembering that Δx and $\delta\hat{x}$ are uncorrelated, we have for Eq. (26)

$$\dot{Y} = (F - G\Lambda)Y + Y(F' - \Lambda'G') + KRK' \tag{29}$$

Using $K = PM'R^{-1}$

$$\dot{Y} = FY + YF' - G\Lambda Y - Y\Lambda'G' + PM'R^{-1}MP \tag{30}$$

Performance Criterion, $\Phi(t_f)$

It is desired to minimize, in some sense, the propellant expended. We choose to minimize the integral of the square root of the variance of the propellant expenditure.

$$\Phi(t_f) = \int_{t_o}^{t} \dot{\Phi}\, dt = \int_{t_o}^{t} \frac{1}{c}\sqrt{2/\pi}\ \sqrt{\mathcal{E}[\delta u'\ \delta u]}\, dt \tag{31}$$

where c is the effective exhaust velocity. Since

$$\delta u = -\Lambda \delta \hat{x}$$

we have

$$\begin{aligned} \mathcal{E}[\delta u'\ \delta u] &= \text{tr}\ \mathcal{E}[\delta u\ \delta u'] \\ &= \text{tr}\ \mathcal{E}[\Lambda \delta \hat{x}\ \delta \hat{x}' \Lambda'] \\ &= \text{tr}\ \Lambda Y \Lambda' \end{aligned} \tag{32}$$

Hence,

$$\dot{\Phi} = \sqrt{\text{tr}\ \Lambda Y \Lambda'}\ \frac{1}{c}\sqrt{2/\pi} \tag{33}$$

$\dot{\Phi}$ is not the average fuel rate because $\mathcal{E}[\sqrt{\delta u'\ \delta u}] \neq \sqrt{2/\pi}\sqrt{\mathcal{E}[\delta u'\ \delta u]}$. It may be shown, however, that $\Phi(t_f)$ is an upper bound for the average fuel expenditure.

Development of the Γ Matrices

In our midcourse guidance problem, the expanded form of

$$\dot{x} = f(x, u) \tag{34}$$

is written

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{\vec{R}} \\ -\frac{\mu x}{r^3}\vec{R} + \frac{\vec{T}}{m} \end{bmatrix} \tag{35}$$

and

$$\delta\dot{x} = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u = F\,\delta x + G\,\delta u \tag{36}$$

or more specifically we use

$$\begin{bmatrix} \delta\dot{x} \\ \delta\ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \frac{\partial \ddot{x}}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta\dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} I \end{bmatrix} [\delta u] \tag{37}$$

where each matrix written here is a 3×3 matrix and the vectors have three components. From this, one can see that the δu vector consists of the three Cartesian coordinates of the thrust vector.

We consider the idealized case in which a correction is performed in terms of an instantaneous change in the velocity corresponding to a large (impulsive) acceleration. The expression for the change in velocity comes from integrating the acceleration over an interval which becomes vanishingly short in the limit.

$$\ddot{R} = \frac{\vec{T}}{m} = \frac{|T|}{m}\frac{\vec{T}}{|T|} \tag{38}$$

where

$$|T| = -c\dot{m} \tag{39}$$

So

$$\begin{aligned} \ddot{R} &= -\frac{c\dot{m}}{m}\frac{\vec{T}}{|T|} \\ &= -\frac{d}{dt}(c \log m)\frac{\vec{T}}{|T|} \end{aligned} \tag{40}$$

Integrating this gives the change in velocity due to the impulsive thrust. The gravitational terms are omitted in Eq. (38) since they become zero in the limit. Hence

$$\dot{R}^{+} - \dot{R}^{-} = -c \log \frac{m^{+}}{m^{-}} \frac{T}{|T|} \tag{41}$$

Thus for an impulsive correction, the thrust components and the velocity increment components are parallet and the magnitude of the velocity change is

$$\Delta v = -c \log \frac{m^{+}}{m^{-}} \tag{42}$$

Now we have

$$\frac{T}{|T|} = \frac{\dot{R}^{+} - \dot{R}^{-}}{\Delta v} \tag{43}$$

In our midcourse problem we consider the simplified case in which

$$\Lambda = k\Gamma \tag{44}$$

So we have the equalities

$$\vec{T} = \delta\vec{u} = \frac{|T|}{\Delta v}(\dot{R}^{+} - \dot{R}^{-}) = -\Lambda\delta\hat{x} = -k\Gamma\delta\hat{x} \tag{45}$$

where the matrix Γ is derived from a single impulse correction policy. A single impulse correction policy may be chosen to null three position miss components at t_f, an arrival time which will be assumed to be fixed. The scalar factor k becomes the control variable of our variational problem, part of which is the factor $\frac{|T|}{\Delta v}$ from Eq. (45) and the other part is some number dependent on how much of the terminal miss distance we wish to eliminate during a correction. The determination of the k history is the subject of the optimization study.

For the purpose of determining the Γ matrix, we assume that the change in velocity at t_o will completely null the estimated terminal "miss"

$$\dot{R}^{+} - \dot{R}^{-} = \Delta\dot{R}(t_o) = -\Gamma\, \delta\hat{x}(t_o) \tag{46}$$

where the "miss" is a linear function of the six deviation components at time t_o. That is, the "current" estimated state deviations at t_o can be propagated ahead to t_f to estimate the terminal miss distance and velocity. The first three components of $\delta\hat{x}(t_f)$ are those to be nulled.

Making use of the state transition matrix $\Phi(t_f, t_o)$ we have

$$\delta x(t_f) = \Phi(t_f, t_o)\, \delta x(t_o) \tag{47}$$

When we partition the transition matrix into four 3×3 matrices

$$\Phi(t_f, t_o) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{48}$$

we can write the desired terminal miss distance, namely zero, in terms of the existing deviations plus the correction to the velocity components as

$$\begin{bmatrix} \delta x(t_f) \\ \delta y(t_f) \\ \delta z(t_f) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = [A]\, \delta x(t_o) + [B][\delta\dot{x}(t_o) + \Delta\dot{R}(t_o)] \tag{49}$$

where $\delta x(t_o)$ and $\delta\dot{x}(t_o)$ are three component vectors.

Solving for $\Delta\dot{R}(t_o)$ from Eq. (49) gives us

$$\Delta\dot{R}(t_o) = -[B^{-1}A]\,\delta x(t_o) - \delta\dot{x}(t_o)$$

$$\underset{(3\times1)}{[\Delta\dot{R}(t_o)]} = -[\underset{(3\times3)}{B^{-1}A} \;\vdots\; \underset{(3\times3)}{I}\;]\;\underset{(6\times1)}{\begin{bmatrix}\delta x(t_o)\\ \delta\dot{x}(t_o)\end{bmatrix}} \tag{50}$$

So for a control policy which permits correction in all three coordinates of the velocity vector.

$$\underset{(3\times6)}{\Gamma} = [\underset{(3\times3)}{B^{-1}A} \;\vdots\; \underset{(3\times3)}{I}\;] \tag{51}$$

If, however, the system were restricted and could function with only one degree of freedom, for instance, if only one component of the velocity could be changed, the form of Γ would be somewhat different. This would correspond to the case of a rocket firing along some specified axis (e.g., spin axis) in inertial space.

The Cartesian coordinates of the additional velocity would be

$$\begin{bmatrix}\Delta\dot{R}_1\\ \Delta\dot{R}_2\\ \Delta\dot{R}_3\end{bmatrix} = \begin{bmatrix}\cos\alpha_o \;\; \cos\delta_o\\ \sin\alpha_o \;\; \cos\delta_o\\ \sin\delta_o\end{bmatrix}\Delta v = N\,\Delta v \tag{52}$$

where α_o and δ_o would be the right ascension and declination respectively of the fixed axis along which the thrust is applied.

Again we write the desired terminal miss in terms of the existing deviations plus the corrections to the velocity components as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = [A]\,\delta x(t_o) + [B][\delta\dot{x}(t_o) + N\,\Delta v] \tag{53}$$

from which we get

$$\underset{(3\times 1)}{BN\,\Delta v} = \underset{(3\times 1)}{-A\,\delta x(t_o)} - \underset{(3\times 1)}{B\,\delta\dot{x}(t_o)} \tag{54}$$

Since we have, in effect, three equations with only one unknown, we apply the method of least squares to obtain the best value of Δv from all three equations. This corresponds to minimizing the miss magnitude. Premultiply both sides of Eq. (54) by the transpose of the coefficient of Δv

$$N'B'BN\,\Delta v = -N'B'[A \vdots B]\begin{bmatrix} \delta x(t_o) \\ \delta\dot{x}(t_o) \end{bmatrix} \tag{55}$$

So

$$\Delta v = -(N'B'BN)^{-1}N'B'[A \vdots B]\begin{bmatrix} \delta x(t_o) \\ \delta\dot{x}(t_o) \end{bmatrix} \tag{56}$$

but we want

$$\Delta\dot{R}(t_o) = -\Gamma\begin{bmatrix} \delta x(t_o) \\ \delta\dot{x}(t_o) \end{bmatrix} = N\,\Delta v \tag{57}$$

Hence

$$\Delta\dot{R}(t_o) = -N(N'B'BN)^{-1}N'B'[A \vdots B]\begin{bmatrix} \delta x(t_o) \\ \delta\dot{x}(t_o) \end{bmatrix} \tag{58}$$

So for one degree of freedom in control

$$\Gamma = N(N'B'BN)^{-1}N'B'[A \vdots B] \tag{59}$$

Finally, there is also the possibility of having only two degrees of freedom in a correction policy. An example might be a rotating disc type of vehicle with its spin axis oriented in inertial space and the direction of thrust or velocity change lying in the plane of the rotating disc. This would require a pulsing type of a rocket firing radially over a small fraction of a revolution of the disc. The two degrees of freedom in the correction policy would then be the magnitude of the velocity vector Δv and the angle β measured from some fixed inertial axis in the disc to the velocity vector. If we had Δv and β given we could get the Cartesian coordinates of the incremental velocity by

$$\begin{bmatrix} \Delta\dot{R}_1 \\ \Delta\dot{R}_2 \\ \Delta\dot{R}_3 \end{bmatrix} = A_3(\alpha_o + 90^o)\, A_1(90^o - \delta_o) \begin{bmatrix} \Delta v \cos\beta \\ \Delta v \sin\beta \\ 0 \end{bmatrix} \tag{60}$$

where $A_3(\theta)$ represents a rotation about the third axis through an angle θ

$$A_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{61}$$

$$A_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{62}$$

The angles α_o and δ_o are the right ascension and declination of the inertial spin axis of the disc.

Evaluating Eq. (60) gives

$$\begin{bmatrix} \Delta\dot{R}_1 \\ \Delta\dot{R}_2 \\ \Delta\dot{R}_3 \end{bmatrix} = \begin{bmatrix} -\sin\alpha_o & -\cos\alpha_o & 0 \\ \cos\alpha_o & -\sin\alpha_o & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\delta_o & -\cos\delta_o \\ 0 & \cos\delta_o & \sin\delta_o \end{bmatrix} \begin{bmatrix} \Delta v\cos\beta \\ \Delta v\sin\beta \\ 0 \end{bmatrix}$$

So

$$\underset{(3\times 1)}{\begin{bmatrix} \Delta\dot{R}_1 \\ \Delta\dot{R}_2 \\ \Delta\dot{R}_3 \end{bmatrix}} = \underset{(3\times 2)}{\begin{bmatrix} -\sin\alpha_o & -\cos\alpha_o\sin\delta_o \\ \cos\alpha_o & -\sin\alpha_o\sin\delta_o \\ 0 & \cos\delta_o \end{bmatrix}} \underset{(2\times 1)}{\begin{bmatrix} \Delta v\cos\beta \\ \Delta v\sin\beta \end{bmatrix}} \tag{64}$$

which we define

$$[\Delta\dot{R}] = C\begin{bmatrix} \Delta v\cos\beta \\ \Delta v\sin\beta \end{bmatrix} \tag{65}$$

Substitute Eq. (65) into Eq. (49). This time we have three equations with two unknowns, namely $\Delta v\cos\beta$ and $\Delta v\sin\beta$.

$$BC\begin{bmatrix} \Delta v\cos\beta \\ \Delta v\sin\beta \end{bmatrix} = -[A \vdots B]\begin{bmatrix} \delta x(t_o) \\ \delta\dot{x}(t_o) \end{bmatrix} \tag{66}$$

Applying the method of least squares to our overdetermined system gives the same form as Eq. (56)

$$\begin{bmatrix} \Delta v\cos\beta \\ \Delta v\sin\beta \end{bmatrix} = -(C'B'BC)^{-1}C'B'[A \vdots B]\begin{bmatrix} \delta x(t_o) \\ \delta\dot{x}(t_o) \end{bmatrix} \tag{67}$$

But, again, we want $\Delta\dot{R}(t_o)$

$$\Delta\dot{R}(t_o) = -\Gamma\begin{bmatrix} \delta x(t_o) \\ \delta\dot{x}(t_o) \end{bmatrix}$$

So

$$\Delta\dot{R}(t_o) = C\begin{bmatrix}\Delta v\cos\beta\\ \Delta v\sin\beta\end{bmatrix} = -C(C'B'BC)^{-1}C'B'[A \vdots B]\begin{bmatrix}\delta x(t_o)\\ \delta\dot{x}(t_o)\end{bmatrix} \tag{68}$$

Finally for two degrees of freedom in control

$$\Gamma = C(C'B'BC)^{-1}C'B'[A \vdots B] \tag{69}$$

Eqs. (51), (59) and (69) respectively we call Γ_3, Γ_1, and Γ_2 corresponding to three, one, and two degrees of freedom respectively.

The Form of Q

The actuation error, v, which appears in Eqs. (2), (4), and (6) could be thought of as some percent error in control which is converted to a $\delta\dot{x}$ variable. Thus, one possible definition would be

$$v = \eta G\,\delta u$$

where η is that percent of the control which is taken to be the error and $G\,\delta u$ is control converted to a $\delta\dot{x}$ variable. Making use of the equation for δu, namely $\delta u = -\Lambda\delta\hat{x}$, we have

$$v = -\eta G\Lambda\delta\hat{x}$$

So the form of Q for such an error would be

$$\begin{aligned}Q(t) &= \mathcal{E}[v(t)\;v'(t)]\\ &= \mathcal{E}[\eta^2 G\Lambda\delta\hat{x}\;\delta\hat{x}'\Lambda'G'] = \eta^2 G\Lambda\mathcal{E}[\delta\hat{x}\;\delta\hat{x}']\Lambda'G'\\ &= \eta^2 G\Lambda Y\Lambda'G'\end{aligned}$$

The Adjoint System

The differential equations governing the covariance matrices and the propellant estimate become the state equations for the midcourse optimization problem:

$$\dot{P} = F P + P F' - P Z P + Q \tag{70}$$

$$\dot{Y} = F Y + Y F' - Y \Lambda' G' - G \Lambda Y + P Z P \tag{71}$$

$$\dot{\Phi} = \sqrt{\frac{2}{\pi}} \operatorname{tr} \Lambda Y \Lambda' \tag{72}$$

where

$$Z = M' R^{-1} M$$

and

$$Q = \eta^2 G \Lambda Y \Lambda' G'$$

The Hamiltonian of the system is

$$H = \dot{\Phi} + \operatorname{tr} (L_p \dot{P} + L_y \dot{Y}) \tag{73}$$

where L_p and L_y are Lagrange multiplier matrices. Note that the Hamiltonian is a scalar and has the usual properties irrespective of the matrix format of the problem.

Returning to the $\dot{P}$, $\dot{Y}$ and $\dot{\Phi}$ equations, we note that

$$\dot{P} = \dot{P}(P, Y, \Lambda, \Lambda')$$

$$\dot{Y} = \dot{Y}(P, Y, \Lambda, \Lambda')$$

$$\dot{\Phi} = \dot{\Phi}(Y, \Lambda, \Lambda')$$

where Φ and the elements of P and Y are the state variables and the elements of Λ and Λ' take on the role of control variables.

Linearization of the system leads to the following form for the variational equations:

$$\begin{aligned}
\delta\dot{P} &= (F - PZ)\,\delta P + \delta P\,(F' - ZP) \\
&\quad + \eta^2[G\Lambda\,\delta Y\,\Lambda' G' + G\,\delta\Lambda\, Y\Lambda' G' + G\Lambda Y\,\delta\Lambda' G'] \\
\delta\dot{Y} &= (F - G\Lambda)\,\delta Y + \delta Y\,(F' - \Lambda' G') - Y\,\delta\Lambda' G' - G\,\delta\Lambda\, Y \\
&\quad + PZ\,\delta P + \delta P\, ZP \\
\delta\dot{\Phi} &= \frac{\mathrm{tr}\,(\Lambda\,\delta Y\,\Lambda' + \delta\Lambda\, Y\Lambda' + \Lambda Y\,\delta\Lambda')}{\sqrt{2\pi\ \mathrm{tr}\,\Lambda Y \Lambda'}}
\end{aligned} \tag{74}$$

We will now evaluate the time derivative of

$$\mathrm{tr}\,(L_p\,\delta P + L_y\,\delta Y) + \delta\Phi \tag{75}$$

and show that the resulting expression is exactly the variation in H, δH due to variations only in the control $\delta\Lambda$ and $\delta\Lambda'$, as long as we define L_p and L_y as solutions of an adjoint system obtained by equating the coefficients of δP and δY, the state variations, to zero, as in Green's formula.

The time derivative of (75) is

$$\mathrm{tr}\,[\dot{L}_p\,\delta P + L_p\,\delta\dot{P}] + \mathrm{tr}\,[\dot{L}_y\,\delta Y + L_y\,\delta\dot{Y}] + \delta\dot{\Phi} \tag{76}$$

Substituting for $\delta\dot{P}$, $\delta\dot{Y}$ and $\delta\dot{\Phi}$ from Eq. (74) and collecting coefficients of δP, δY, $\delta\Lambda$ and $\delta\Lambda'$ making use of the cyclic rule for trace operations, we obtain

$$\mathrm{tr}\,[\dot{L}_p + L_p(F - PZ) + (F' - ZP)L_p + L_y PZ + ZPL_y]\,\delta P$$

$$+ \mathrm{tr}\,[\dot{L}_y + L_y(F - G\Lambda) + (F' - \Lambda' G')L_y$$

$$+ \frac{\Lambda'\Lambda}{\sqrt{2\pi\ \mathrm{tr}\ \Lambda Y \Lambda'}} + \eta^2 \Lambda' G' L_p G \Lambda]\,\delta Y$$

$$+ \mathrm{tr}\,[L_p \eta^2 G\,\delta\Lambda Y \Lambda' G' + L_p \eta^2 G \Lambda Y\,\delta\Lambda' G']$$

$$+ \mathrm{tr}\,[L_y(-Y\,\delta\Lambda' G' - G\,\delta\Lambda Y)] + \frac{\mathrm{tr}\,[\delta\Lambda Y \Lambda' + \Lambda Y\,\delta\Lambda']}{\sqrt{2\pi\ \mathrm{tr}\ \Lambda Y \Lambda'}} \tag{77}$$

Setting the coefficients of δP and δY to zero, we obtain the following differential equations which define the adjoint system:

$$\begin{aligned} \dot{L}_p &= -L_p(F - PZ) - (F' - ZP)L_p - L_y PZ - ZPL_y \\ \dot{L}_y &= -L_y(F - G\Lambda) - (F' - \Lambda' G')L_y - \frac{\Lambda'\Lambda}{\sqrt{2\pi\ \mathrm{tr}\ \Lambda Y \Lambda'}} \\ &\quad - \eta^2 \Lambda' G' L_p G \Lambda \end{aligned} \tag{78}$$

It is to be noted that the remaining terms of (77) is the variation in H, δH due to variations in the controls $\delta\Lambda$ and $\delta\Lambda'$, as can be verified by the reader by taking the variation of the expression for H given by Eq. (73). Thus

$$\int_{t_o}^{t_f} \delta H\ dt = [\mathrm{tr}\,(L_p\,\delta P + L_y\,\delta Y) + \delta\Phi]_{t_o}^{t_f} \tag{79}$$

An appropriate generalization of this expression to the case of strong control variations is given by

$$\int_{t_o}^{t_f} (H^* - H)\,dt = [\mathrm{tr}\ L_p\,\delta P + L_y\,\delta Y + \delta\Phi]_{t_o}^{t_f} \tag{80}$$

as developed in Ref. 5. It then follows that the controls minimizing H are necessary (although not sufficient) for a minimum.

Actuation Errors Independent of Correction Magnitude

If the system employed for implementation of the guidance law contains errors whose covariances are independent of the magnitude of the correction, as well as those described by Eq. (70) with proportional covariances, the matrix Q will then take the form

$$Q = Q_1 + Q_2$$

The second member, Q_2, will be of the form

$$Q_2 = \bar{Q}\, u(k)$$

in which $\bar{Q}$ is a constant matrix and the scalar function $u(k)$ is the unit step function which takes on the value unity during a correction and the value zero otherwise.

Because Q_2 does not possess a proper differential, the equations of variation are not applicable in the case of such errors. Since $\bar{Q}\, u(k)$ is independent of the state, however, the definition of the adjoint system (78) does not require alteration. The appropriate Hamiltonian function for the optimal guidance problem is that given by (73), it being understood that the term Q_2 is included in $\dot{P}$. It should be noted that the function H depends upon the control variable k nonlinearly only because of the actuation error terms Q_1 and Q_2.

The Hamiltonian

The Hamiltonian function which is to be minimized at each t in the interval $t_o \leq t \leq t_f$, by the optimal control is

$$H = \mathrm{tr}\,(L_p \dot{P} + L_y \dot{Y}) + \sqrt{\frac{2}{\pi}\,\mathrm{tr}\,\Lambda Y \Lambda'} \tag{81}$$

We consider the simplified case

$$\Lambda = k\Gamma \tag{82}$$

in which the matrix of feedback parameters Λ is obtained as the matrix Γ corresponding to an appropriate single impulse correction policy, except for a scalar factor k, assumed nonnegative, which becomes the control variable of our variational problem. By substituting the values for $\dot{P}$ and $\dot{Y}$ from Eq. (70) and (71) and $\Lambda = k\Gamma$, into Eq. (81), the Hamiltonian can be expressed as

$$H = ak + bk^2 + cu(k) + d \qquad \begin{array}{l} u(k) = 1 \text{ when } k > 0 \\ u(k) = 0 \text{ when } k = 0 \end{array} \tag{83}$$

where

$$\begin{aligned} a &= \sqrt{\frac{2}{\pi}\,\mathrm{tr}\,\Gamma Y \Gamma'} - \mathrm{tr}\,[L_y G \Gamma Y + L_y Y \Gamma' G'] \\ b &= \mathrm{tr}\,[\eta^2 L_p G \Gamma Y \Gamma' G'] \\ c &= \mathrm{tr}\,[L_p \bar{Q}] \\ d &= \mathrm{tr}\,[L_p (FP + PF' - PZP) + L_y (FY + YF' + PZP)] \end{aligned} \tag{84}$$

and b may be anticipated as positive for minimum fuel problems.

In order that our payoff function be minimized, the maximum principle states that the optimal control $k(t)$ minimize the Hamiltonian for all t, $t_o \le t \le t_f$, or that

$$H[k^*(t)] - H[k(t)] \ge 0 \qquad k, k^* \ge 0$$

A Proposed Optimization Technique

In the following we present an optimization scheme tailored to the features peculiar to the midcourse guidance problem. The sequence of computations would proceed as follows. The first reference solution of the state system $\dot{P}$, $\dot{Y}$, and $\dot{\Phi}$ would be integrated forward with $k(t) = 0$, thus generating the covariance matrices for the case of no guidance corrections. The adjoint system would then be integrated backwards from the terminal point to the initial with terminal conditions on the variables L_p and L_y depending on a provisional choice of a parameter λ, to be discussed later. During this backward integration, the function $H(k^*)$ would be examined at each integration point.

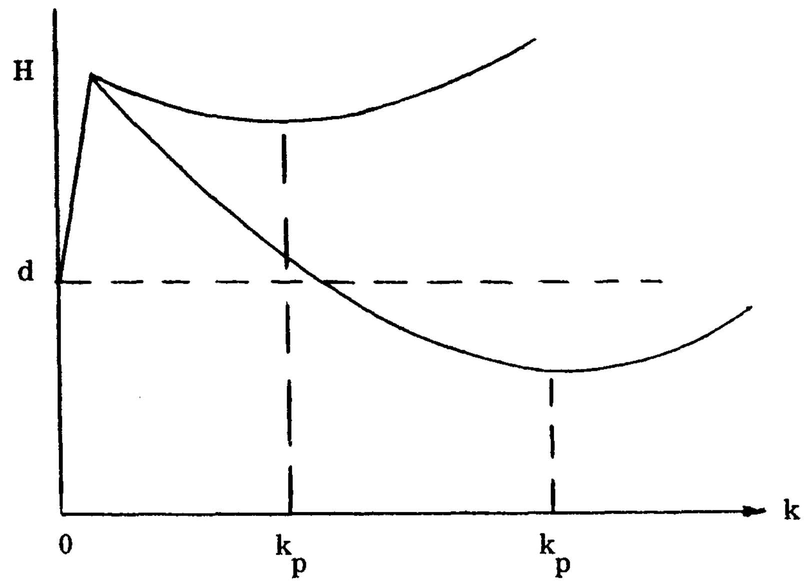
From Eq. (83), the form of H versus k^* is that of a parabola superposed on a function having a finite discontinuity at $k^* = 0$ arising from the appearance of the step function

$$u(k) = 1 \quad \text{when} \quad k > 0$$

and

$$u(k) = 0 \quad \text{when} \quad k = 0$$

Fig. 1



We wish to find that value of $H(k_p)$ corresponding to the minimum of the parabola, which may or may not be less than the value of $d = H(0)$, see Fig. 1.

Now

$$\frac{\partial H}{\partial k} = a + 2bk_p = 0 \qquad \text{for} \qquad H(k_p) = H_p \tag{85}$$

and

$$k_p = -\frac{a}{2b} \tag{86}$$

or

$$H_p = \frac{ak_p}{2} + c + d \qquad \text{since} \qquad \begin{matrix} u(k) = 1 \\ k \neq 0 \end{matrix} \tag{87}$$

If $k_p > 0$, the value is a candidate and we now wish to compare the value of $H(k_p)$ attained at the minimum of the parabola, with $H(0) = d$, so as to decide the global minimum of H. If $k_p < 0$, $k = 0$ will be the value placed in the table since negative k is inadmissible by assumption. Then the iteration process is, at each integration step,

(1) Verify that $b > 0$, since $\frac{d^2H}{dk^2} = 2b$.

(2) Test if $k_p < 0$. If so, $k = 0$ replaces value in the table.

(3) If $k_p > 0$, compute $\Delta H = H_p - d$. If $\Delta H \geq 0$, $k = 0$ gives min H at t, and $k = 0$ replaces value in the table. If $\Delta H < 0$, k_p gives min H at t, and k_p will replace the value in the table. A new solution of the state system is then calculated and the new (and presumably improved) value of the payoff function computed at the terminal point.

The coefficients a, b, c and d of H(k) will, of course, change due to new tables of k(t), generated by the above procedure. For each new table of k(t) steps 1-3 are repeated until convergence.

Boundary and Transversality Conditions

Initial conditions for a typical problem of midcourse guidance along a free fall space trajectory are

$$P(t_o) = P_o \tag{88}$$

$$Y(t_o) = 0 \tag{89}$$

where P_o is the covariance matrix of injection errors determined by a priori analysis of the injection guidance system.

If all three components of position miss at a fixed final time are to be corrected, a measure of the magnitude of miss variance such as

$$\xi(t_f) = y_{11}(t_f) + y_{22}(t_f) + y_{33}(t_f) + p_{11}(t_f) + p_{22}(t_f) + p_{33}(t_f) \tag{90}$$

may be specified. If the function of terminal values

$$\Phi(t_f) + \lambda_\xi \xi(t_f) \tag{91}$$

is adopted as the function to be minimized, the constant multiplier $\lambda_\xi > 0$ has the interpretation $-\dfrac{\partial \Phi(t_f)}{\partial \xi(t_f)}$.

The appropriate transversality condition for the terminal values of the adjoint elements is

$$\mathrm{tr}\,(L_p \delta P + L_y \delta Y) = \lambda_\xi \delta\xi(t_f) \tag{92}$$

which yields

$$\ell_{p_{11}} = \ell_{p_{22}} = \ell_{p_{33}} = \lambda_\xi \tag{93}$$

$$\ell_{y_{11}} = \ell_{y_{22}} = \ell_{y_{33}} = \lambda_\xi \tag{94}$$

and all other $\ell_{p_{ij}}$ and $\ell_{y_{ij}}$ zero.

Selection of the Multiplier λ

Suppose we are interested in that value of λ such that only one nonzero control appears in the table, and $k = 0$ for all other times. This is equivalent to finding the value of λ, $\bar{\lambda}$, which makes ΔH equal to zero at each time. This collection of $\bar{\lambda}$'s then represent the threshold values for nonzero controls at each time. The minimum value of $\bar{\lambda}$ in this collection is then the overall threshold λ, λ_M, we are seeking.

In $H = ak + bk^2 + cu(k) + d$ and from Eq. (84), which defines a, b, c and d and from the transversality conditions of Eqs. (93) and (94), we notice that H is linear in λ, that is, H is of the form

$$H = \dot{\Phi}(k) + h(k)\lambda \tag{95}$$

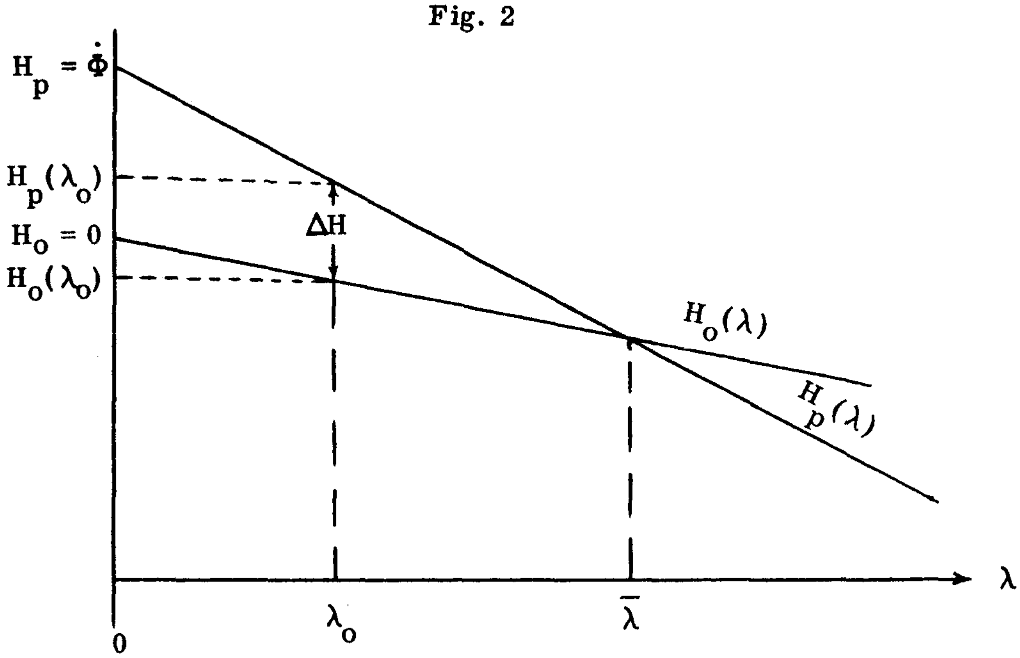
where $\dot{\Phi}$ and h are functions of k. For each time, for $k = 0$

$$H(\lambda) = H_o(\lambda) = d = N\lambda \tag{96}$$

and for $k = k_p$

$$H(\lambda) = H_p(\lambda) = \dot{\Phi} + M\lambda \tag{97}$$

Fig. 2



From Fig. 2, M and N are the slopes for H_p and H_o respectively. For any λ_o, $H_p(\lambda_o)$ and $H_o(\lambda_o)$ can be computed.

At $\lambda = 0$,

$$H_p = \dot{\Phi}(k_p)$$

$$H_o = 0$$

The slopes are

$$M = \frac{-[\dot{\Phi}(k_p) - H_p(\lambda_o)]}{\lambda_o} \tag{98}$$

$$N = \frac{H_o(\lambda_o)}{\lambda_o} \tag{99}$$

We wish to find $\bar{\lambda}$ for which

$$H_p(\bar{\lambda}) = H_o(\bar{\lambda}) \tag{100}$$

Substituting Eqs. (96) and (97) into condition (100) yields,

$$\dot{\Phi} + M\bar{\lambda} = N\bar{\lambda}$$

and

$$\bar{\lambda} = \frac{\dot{\Phi}}{N - M}$$

and finally

$$\bar{\lambda} = \frac{\dot{\Phi}\,\lambda_o}{\dot{\Phi} - \Delta H(\lambda_o)} \tag{101}$$

is the $\bar{\lambda}$ at each time from which λ_M will be chosen by selection of the smallest. This value of λ_M will be employed for a generation of our first optimal control history. A family of optimal control histories, requiring a greater number of control commands will then be generated by selecting larger values for λ.

REFERENCES

1. Denham, W.F. and Speyer, J.L.; Optimal Measurement and Velocity Correction Programs for Midcourse Guidance, AIAA Journal, May 1964.

2. Striebel, C.T. and Breakwell, J.V.; Minimum Effort Control in Interplanetary Guidance, IAS, January 1963.

3. Kalman, R.E. and Bucy, R.S.; New Result in Linear Filtering and Prediction Theory, Journal of Basic Engineering, Transactions of ASME, Vol. 83, March 1961.

4. Kelley, H.J., Kopp, R.E. and Moyer, H.G.; Successive Approximation Techniques for Trajectory Optimization, IAS Vehicle Systems Optimization Symposium, Garden City, N.Y., November 1961.

5. Kopp, R.E.; Pontryagin Maximum Principle, Chapter 7 of "Optimization Techniques," G. Leitmann, Editor, Academic Press, New York, 1962.